

Bounded Control Mobile Robot Navigation Functions

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Abstract—*Navigation functions combine geometric path planning and control of mobile robots in the presence of obstacles. This paper extends the theory of mobile robot navigation functions by proposing a navigation function feedback control law that ensures safe navigation of a point mass robot under bounded control inputs. Simulations describe the robot state variables navigating under bounded control in planar environments populated by disc obstacles.*

I. INTRODUCTION

Navigation functions combine geometric path planning with the ability to control mobile robots in a manner that ensures collision safety and arrival to the target. Navigation functions have the following essential properties for robotic navigation [12] :

- They are *polar* with a unique minimum at the target while all other critical points are non-degenerate saddles.
- They are *admissible* with a uniformly maximal height on the obstacle boundaries.

Navigation functions are discussed with examples in [4][Chap. 4], [13][Chap. 8] and [19][Chap. 10]. There have been successful navigation function applications to problems of increasing complexity such as multi-robot systems [6], [10], [16], [22], dynamic environments [2], [3] and mobile robot manipulation tasks [23].

One of the open challenges in mobile robot navigation function theory is the need to ensure bounded control inputs while preserving the navigation function collision safety and arrival to the target properties. This paper describes a novel navigation function based bounded control law for a point mass robot moving in a planar environment populated by disc obstacles. While not the focus of this paper, disc worlds serve as model spaces for general planar environments via appropriate transformations [12], [17].

Related literature: Since its introduction [12], advances in robot navigation functions include on-line tuning of navigation functions by Filippidis [7] and Lionis [14], and sensor based navigation functions by Arslan [1], Vasilopoulos [24] and Loizou [18]. Tanner expanded the applicability of navigation functions to control barrier functions under temporal logic predicates [26] as well as to navigation functions in the presence of moving obstacles [2], [25]. Rousseas expanded navigation functions to include

path optimality criteria as well as navigation functions approximated by reinforcement learning techniques [20].

Harmonic navigation functions originally proposed by [5], [11], [21] were used by Loizou [15] to provide a simple *tuning law*, defined as the selection of a parameter k that represents the relative magnitude of the attraction-to-the-target component compared to the obstacle-repulsion component in the navigation function. When k is larger than the number of obstacles in the environment, all spurious local minima become non-degenerate saddles and the potential function becomes polar at the target [18]. This tuning law is used in the current paper.

Paper contributions: The paper considers a point mass robot navigating under bounded control in a planar disc world. The paper describes a closed loop navigation function control law that ensures safe navigation under bounded control inputs, demonstrating a feasible bounded control policy that maintains the navigation function collision safety and arrival to the target properties. Simulation studies then describe the robot state variables navigating under bounded control inputs in planar disc worlds.

The paper is structured as follows. Sections 2 and 3 summarize the notion of robot navigation functions and their construction on disc worlds. Sections 4 and 5 describe the bounded control navigation function control law, then simulation studies of the bounded control law in planar disc worlds. The conclusion summarizes the results and discusses future research topics.

II. NAVIGATION FUNCTIONS ON DISC WORLDS

This section defines the robot model and the disc worlds, navigation functions on disc worlds, then a statement of the bounded control input navigation problem considered in this paper.

The paper considers an omnidirectional point robot with mass m and second-order dynamics

$$m \cdot \ddot{x}(t) = u(t) \quad u(t) \in \mathbb{R}^2$$

where the robot position and velocity, $(x, \dot{x}) \in \mathbb{R}^2 \times \mathbb{R}^2$, form its state variables. The control input $u(t)$ is piecewise continuous and importantly must satisfy the bound

$$\|u(t)\| \leq F_{max} \quad (1)$$

where $\|\cdot\|$ denotes Euclidean norm and F_{max} is a system actuation limit. The point robot is assumed to measure its current state using an ideal position and velocity sensor.

Disc worlds: The point robot moves in a disc world consisting of a large disc, D_0 , whose interior is populated by smaller disjoint disc obstacles $D_1, \dots, D_M$. The robot

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workspace is defined as the compact and connected set $\mathcal{W} = D_0 - \bigcup_{i=1}^M \text{int}(D_i)$, where $\text{int}(D_i)$ denotes the i 'th disc interior.

Navigation function: Let x_T be a target point in the interior of $\mathcal{W}$. A $C^{(2)}$ -smooth function $\varphi : \mathcal{W} \rightarrow [0, 1]$ forms a *navigation function* if:

- 1) φ is polar on $\mathcal{W}$ with a unique minimum at x_T , such that $\varphi(x_T) = 0$.
- 2) All critical points of φ except the minimum at x_T are non-degenerate saddle points.
- 3) φ is admissible on $\mathcal{W}$ with a uniformly maximal height on the obstacle boundaries.

Navigation function control law: The basic navigation function control law is simply the negated gradient of φ augmented by damping proportional to the robot velocity

$$u(t) = -\nabla\varphi(x(t)) - C\dot{x}(t) \quad (2)$$

where C is a positive definite 2×2 damping matrix. Asymptotic convergence of the robot to the target is verified by its total mechanical energy Lyapunov function

$$V(x, \dot{x}) = \varphi(x) + \frac{1}{2}m \cdot \|\dot{x}\|^2$$

whose time derivative along the robot trajectories

$$\frac{d}{dt}V(x, \dot{x}) = \nabla\varphi(x) \cdot \dot{x} + m \cdot (\dot{x} \cdot \ddot{x}) = -\dot{x}^T C \dot{x} \leq 0$$

implies that all robot trajectories convergence to zero velocity states. When augmented with LaSalle's invariance principle, all robot trajectories convergence to zero velocity states at one of the critical points of φ . Since φ has a unique minimum at the target while all other critical points are non-degenerate saddles, the robot reaches the target from almost all initial points in $\mathcal{W}$. Collision safety is ensured when the robot starts with zero speed in $\mathcal{W}$. The admissibility of φ on $\mathcal{W}$ implies that $\varphi(x(t)) \leq V(x(t))$, $\dot{x}(t) \leq 1$ for all $t \geq 0$, thus ensuring that the robot remains within its workspace for all $t \geq 0$.

Problem statement: The navigation law of Eq. (2) ignores the need to ensure bounded control input, which is the topic of this paper. Given a navigation function $\varphi : \mathcal{W} \rightarrow [0, 1]$, modify the navigation law of Eq. (2) in a manner that preserves collision safety and arrival to the target under the control input bound $\|u(t)\| \leq F_{max}$.

III. TUNED NAVIGATION FUNCTION CONSTRUCTION

This section describes a simple navigation function construction on disc worlds [12]. The obstacle boundaries are encoded by the following *obstacle functions*. The robot workspace is centered at the origin with radius ρ_0 . The internal disc obstacles are centered at the points $x_1, \dots, x_M$ with radii $\rho_1, \dots, \rho_M$. The obstacle functions are defined as $\beta_0(x) = \rho_0^2 - \|x\|^2$ and $\beta_i(x) = \|x - x_i\|^2 - \rho_i^2$ for $i = 1 \dots M$. The product $\beta(x) = \prod_{i=0}^M \beta_i(x)$ is strictly positive in the workspace interior and vanishes on the workspace boundary.

The *target function* is defined as $\gamma(x) = \|x - x_T\|^2$. Using a tuning parameter k and the obstacle functions product β ,

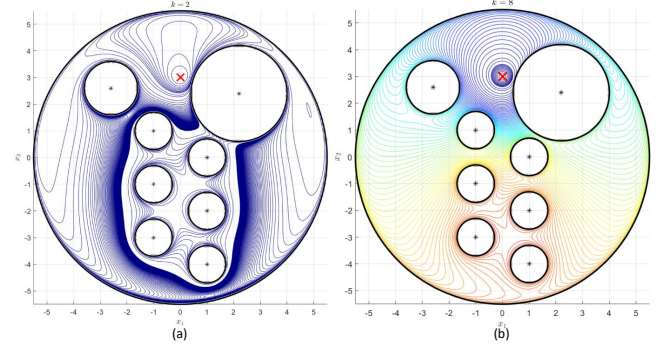


Fig. 1: (a) Contours of φ for $k = 2$ showing a spurious local minimum on the upper right side. (b) When $k = 8$, all spurious local minima disappear and the target (marked as red X) becomes the unique minimum of φ .

a disc world navigation function is given by the function composition (denoted $\circ$)

$$\varphi(x) = \frac{z}{1+z} \circ \frac{\gamma^k(x)}{\beta(x)} = \frac{\gamma^k(x)}{\gamma^k(x) + \beta(x)}$$

where k must be sufficiently large to ensure that all spurious local minima of φ disappear. Note that φ attains values in the unit interval such that $\varphi(x_T) = 0$ and $\varphi(x) = 1$ on the obstacle boundaries.

Example: Fig. 1 shows a disc world containing eight internal obstacles. When $k = 2$, a parasitic local minimum of φ appears on the right side of the corridor formed by the vertically aligned discs (Fig. 1(a)). When the tuning parameter increases to $k = 8$ (see [8] and [17] for a discussion of this tuning criterion), all spurious local minima of φ disappear and the target becomes the unique minimum of φ (Fig. 1(b)). Note that each internal obstacle now incurs a nearby saddle point. $\circ$

IV. BOUNDED INPUT CONTROL LAW

This section describes a closed loop navigation function control law which maintains bounded control inputs. The control law scales the navigation function gradient and splits the damping term in Eq. (2) into two components, aligned with the navigation function gradient and its orthogonal direction.

Let $u \cdot v$ denote vector scalar product and let $\hat{v}$ denote the robot unit magnitude velocity vector, $\hat{v} = \dot{x} / \|\dot{x}\|$ when $\dot{x}$ is non-zero. The control input bound F_{max} of Eq. (1) is replaced by the conservative bounds $2F_T$ and F_N

$$|u(t) \cdot \widehat{\nabla\varphi}(x)| \leq 2F_T \quad \text{and} \quad |u(t) \cdot \widehat{\nabla\varphi}^\perp(x)| \leq F_N \quad (3)$$

where $(2F_T)^2 + F_N^2 \leq F_{max}^2$ and $\widehat{\nabla\varphi}$ and $\widehat{\nabla\varphi}^\perp$ are unit magnitude vectors along the navigation function gradient and its orthogonal direction. The bounded control navigation function law is given by

$$u(t) = -h \cdot \nabla\varphi(x) - F_T (\widehat{\nabla\varphi} \cdot \hat{v}) \widehat{\nabla\varphi}(x) - F_N (\widehat{\nabla\varphi}^\perp \cdot \hat{v}) \widehat{\nabla\varphi}^\perp(x) \quad (4)$$

where h is a positive scaling factor described below. The control law can be written in a manner that highlights its damping matrix

$$u(t) = -h \cdot \nabla \varphi(x(t)) - C(x(t)) \hat{v}(t)$$

where $C = [\widehat{\nabla \varphi} \ \widehat{\nabla \varphi}^\perp] \begin{bmatrix} F_T & 0 \\ 0 & F_N \end{bmatrix} [\widehat{\nabla \varphi} \ \widehat{\nabla \varphi}^\perp]^T$ is positive definite and depends on the robot current position.

To verify collision safety and arrival to the target, consider the modified Lyapunov function

$$V(x, \dot{x}) = h \cdot \varphi(x) + \frac{1}{2} m \cdot \|\dot{x}\|^2$$

whose time derivative along the robot trajectories is

$$\begin{aligned} \frac{d}{dt} V(x, \dot{x}) &= h \cdot \nabla \varphi(x) \cdot \dot{x} + m \cdot (\dot{x} \cdot \ddot{x}) \\ &= -F_T \cdot \|\dot{x}\| (\widehat{\nabla \varphi} \cdot \hat{v})^2 - F_N \cdot \|\dot{x}\| (\widehat{\nabla \varphi}^\perp \cdot \hat{v})^2 \leq 0. \end{aligned}$$

It follows that all robot trajectories converge to zero velocity states at one of the critical points of φ . Since φ has a unique minimum at the target while all other critical points are non-degenerate saddles, the robot reaches the target from almost all initial points in $\mathcal{W}$. When the robot starts with zero speed in $\mathcal{W}$, the admissibility of φ on $\mathcal{W}$ implies that $h \cdot \varphi(x(t)) \leq V(x(t), \dot{x}(t)) \leq h$, thus ensuring collision safety for all $t \geq 0$.

Next consider the control input bounds of Eq. (3). Projecting $u(t)$ of Eq. (4) along $\widehat{\nabla \varphi}^\perp$ gives

$$|u(t) \cdot \widehat{\nabla \varphi}^\perp| = F_N \cdot |\widehat{\nabla \varphi}^\perp \cdot \hat{v}| \leq F_N$$

as required by Eq. (3). Projecting $u(t)$ of Eq. (4) along $\widehat{\nabla \varphi}$ gives

$$|u(t) \cdot \widehat{\nabla \varphi}| = |h \cdot \|\nabla \varphi\| + F_T \cdot (\widehat{\nabla \varphi} \cdot \hat{v})| \leq h \cdot \|\nabla \varphi\| + F_T.$$

Since $|u(t) \cdot \widehat{\nabla \varphi}|$ must be bounded by $2F_T$, the scaling factor h is selected as

$$h = \frac{F_T}{\max_{x \in \mathcal{W}} \{\|\nabla \varphi(x)\|\}} \quad (5)$$

as required by Eq. (3).

Scaling factor caveat: The selection rule for h sheds light on the need to scale the navigation function gradient within the control input bound using global information, $\max_{x \in \mathcal{W}} \{\|\nabla \varphi(x)\|\}$. This computation is currently done numerically over the environment, as illustrated in Fig. 2 which shows the contours of $\|\nabla \varphi\|$ over the entire workspace. See also paper conclusion.

V. SIMULATION RESULTS

This section evaluates the point robot trajectories under the bounded amplitude navigation function law of Eq. (4). The simulations integrate the robot closed loop dynamics using Matlab's *ode45()* command [9]. The simulations use the following parameters

$$m = 1 [kg] \quad k = 4 \quad h = 2.02 \quad F_{max} = 3.354 [N].$$

For the basic control law of Eq. (2), damping is based on the critical damping [18], which prevents oscillations when approaching the target and dampens other unwanted

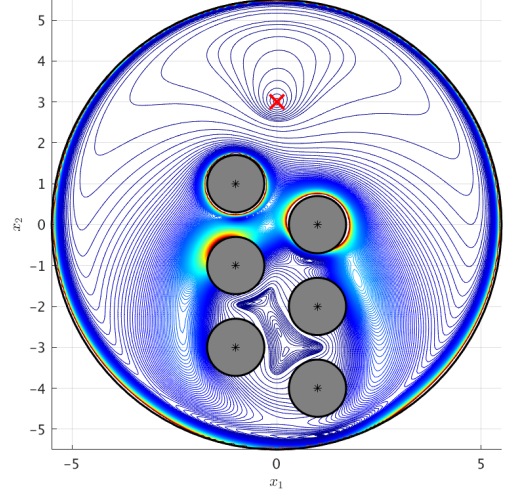


Fig. 2: Contours of $\|\nabla \varphi\|$ over the robot workspace, using $k=4$. The selection rule for h in Eq. (5) requires an upper bound on $\|\nabla \varphi\|$ over the entire robot workspace. Highest values colored yellow and red reach $\|\nabla \varphi\| = 0.74$ in this example.

oscillations during navigation. The damping matrix used in the the basic control law is $C = 3c^* \cdot I$, where I is the identity matrix and $c^* = 0.22 \frac{Ns}{m}$ is the system critical damping when approaching the target [18].

The environment size in all examples is set as $\rho_0 = 5.5 m$ and all robot trajectories terminate within a distance of 1.0 cm from the target.

Example 1: Fig. 3 shows a disc world containing six internal obstacles. Two trajectories of the point robot under the basic control law of Eq. (2) are shown in Fig. 3(a). The trajectories start at slightly different initial points, $x_{S,1}$ (blue) and $x_{S,2}$ (red), to highlight the influence of the nearby saddles. Using the control input bounds $2F_T = F_N = 1.5 N$ and the same initial points, Fig. 3(b) shows two trajectories of the point robot under the bounded amplitude control law of Eq. (4). Fig. 4 depicts the point robot velocity while Fig. 5 depicts the force input $u(t)$ under the bounded amplitude control law of Eq. (4) $\circ$

Example 2: Fig. 6 shows the point robot trajectories for other values of F_T and F_N under the bounded amplitude control law of Eq. (4). Fig. 6(a) shows the point robot trajectories for $2F_T = F_N = 0.5 N$ while Fig. 6(b) shows the trajectories for $2F_T = F_N = 5.0 N$. Fig. 6(c) depicts the point robot velocity while Fig. 7 depicts the force input $u(t)$ under the bounded amplitude control law of Eq. (4). The different control input bounds have minimal influence on the robot trajectories. However, the lower bound of 0.5 N incurred under-damped path oscillations at distances of about 1.0 mm from the target. $\circ$

The control law of Eq. (4) is valid under arbitrary control input bound $F_{max} > 0$. However, the robot system takes longer time to reach the target under the lower control input bound

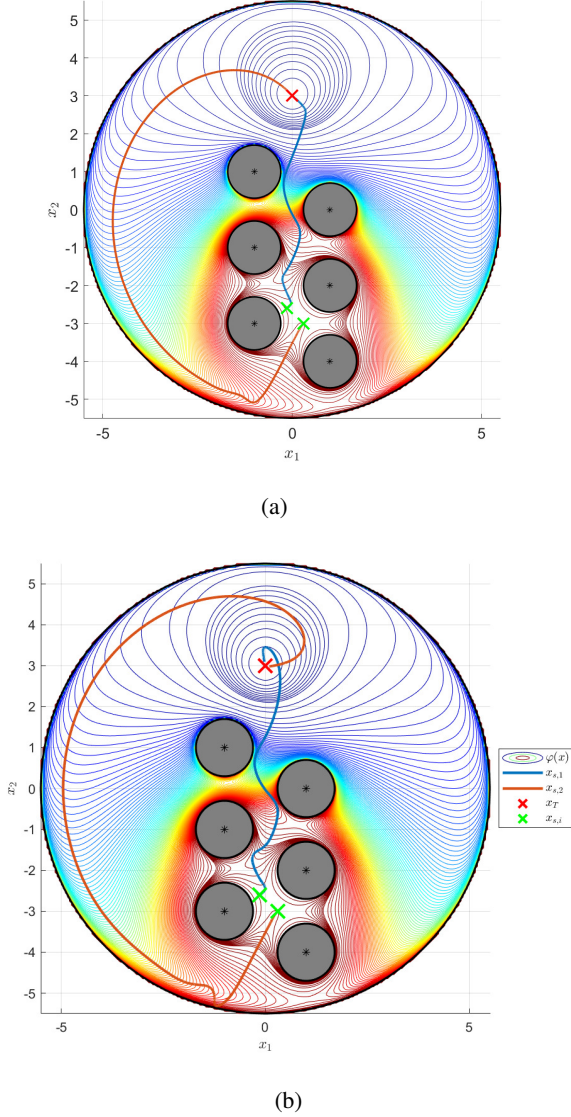


Fig. 3: (a) Trajectories of the closed loop point robot system under the basic law of Eq. (2) using over-damped term. (b) Trajectories of the closed loop point robot system under the bounded amplitude law of Eq. (4). The trajectories start at slightly different initial points, $x_{s,1}$ (blue) and $x_{s,2}$ (red), to highlight the influence of the nearby saddles.

(Figs. 5 and 7).

Example 3: The final example shows in Fig. 8 a disc world containing eight internal obstacles whose position and size were randomly selected. Using the control input bounds $2F_T = F_N = 1.5$ N, Fig. 8 shows two trajectories of the point robot under the bounded amplitude control law of Eq. (4). $\circ$

VI. CONCLUSION

The paper described how mobile robot navigation functions can be used in environments populated by disc obstacles when the robot operates under bounded control inputs. This important feature has been missing from current mobile robot

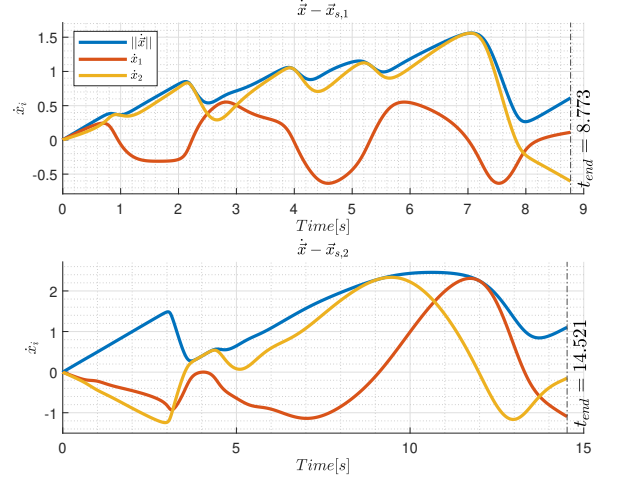


Fig. 4: The point robot velocity along the trajectories of Fig. 3(b) under the bounded amplitude control law.

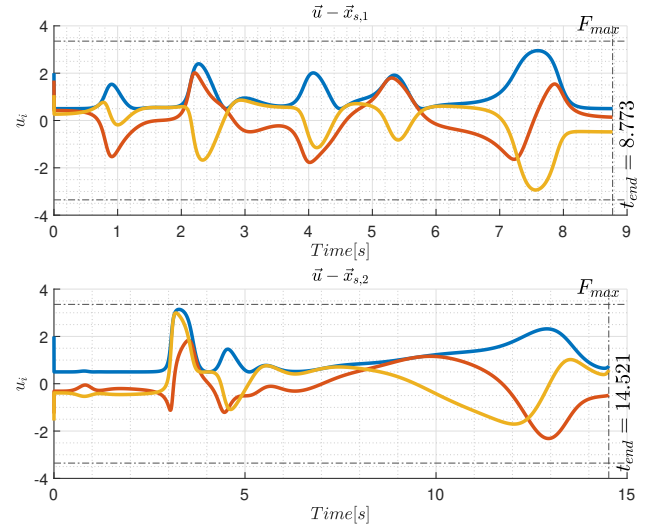


Fig. 5: The point robot control force input $u(t)$ along the trajectories of Fig. 3(b) under the bounded amplitude control law.

navigation theory.

The proposed control law scales the navigation function gradient and splits the damping term into components aligned with the navigation function gradient and its orthogonal direction. Using conservative control input bounds, $2F_T$ along the navigation function gradient and F_N along its orthogonal direction, the paper established a rule for selecting the navigation function gradient scaling factor, $h = \max_{x \in \mathcal{W}} \{ \|\nabla \varphi(x)\| \}$. Simulations of the closed loop robot system illustrate the robot trajectories as well as the evolution of its state and control inputs during navigation in planar disc worlds under bounded control inputs.

A major caveat with the proposed navigation function gradient scaling is the need to compute an upper bound

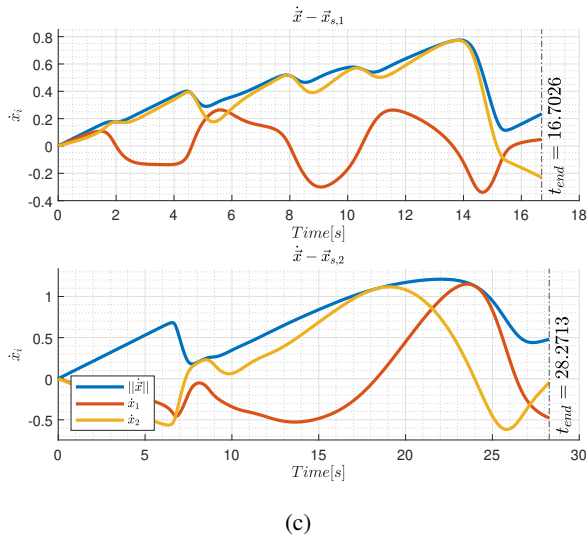
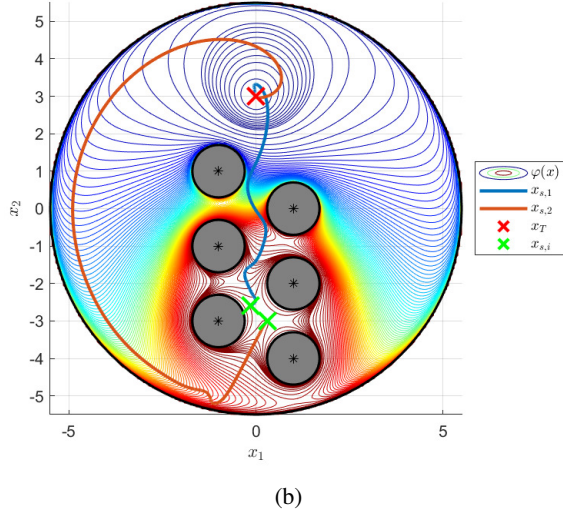
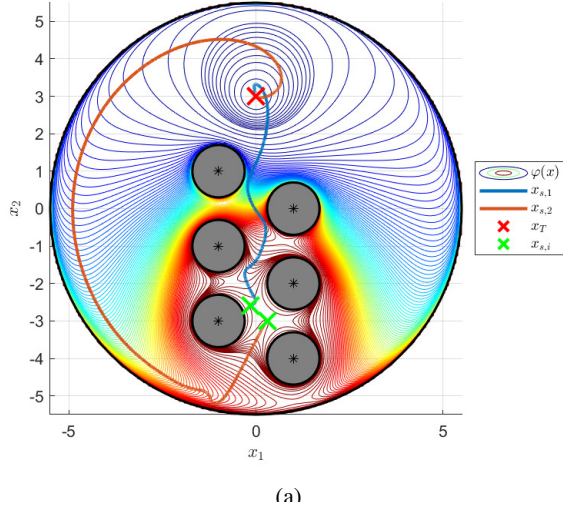


Fig. 6: The point robot trajectories of Example 2, under the bounded amplitude control law: (a) $2F_T = F_N = 0.5$ N, (b) $2F_T = F_N = 5.0$ N. (c) The point robot velocity along the trajectories of Fig. 6(a) under the control input bounds $2F_T = F_N = 0.5$ N.

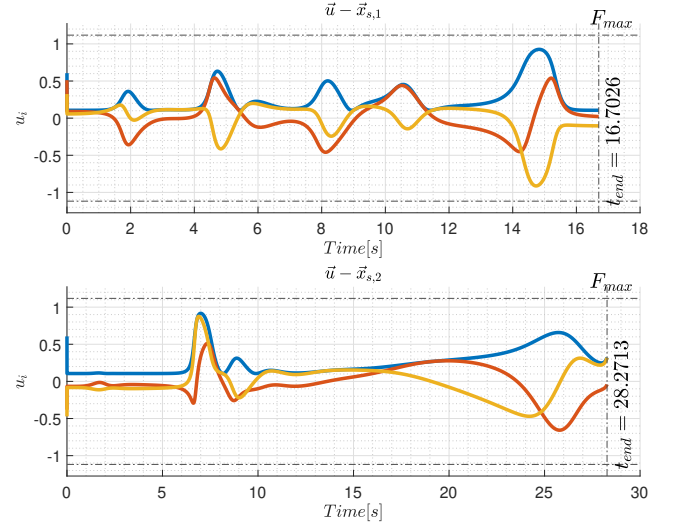


Fig. 7: The point robot control force input $u(t)$ along the trajectories of Fig. 6(a) for the control input bounds $2F_T = F_N = 0.5$ N.

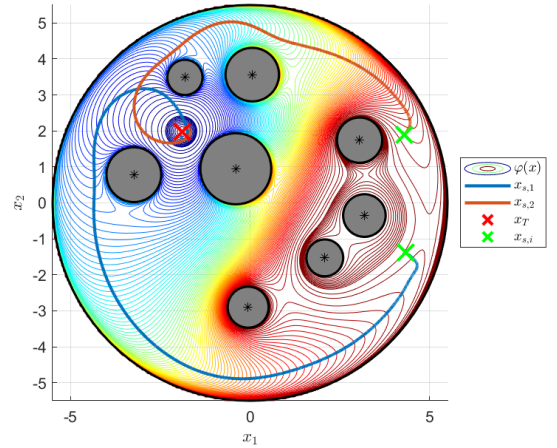


Fig. 8: Two trajectories of the point robot system under the bounded amplitude law of Eq. (4) in a randomly selected disc-world containing eight internal obstacles.

on $\|\nabla\varphi\|$ over the entire robot workspace. This paper used numerical evaluation of this bound. However, disc worlds serve as model spaces for general planar environments. When used as model spaces, the disc world navigation function is composed with a coordinate transformation that maps the true environment to its model disc world. Consequently, the navigation function gradient is modulated by the coordinate transformation Jacobian which reflects the true control input needed to avoid obstacles in the environment. The efficient estimation of this quantity is thus a major challenge that needs to be explored in future research.

Our longer term research seeks to incorporate explicit amplitude modulating control inputs into the navigation func-

tion based control law. Using amplitude modulating control inputs, navigation functions will fit into an optimal control framework. The navigation function obstacle avoidance and arrival to the target properties augmented with control input bounds will become state dependent constraints on the control inputs. Now with the ability to add path optimality criteria such as time optimality or energy expenditure optimality.

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